Enumerative Geometry of CY 4-folds

— what happens in ideal world... —

multiple covers formula & meeting invariants

Klemm, Pandharipande

Gromov–Witten invariants ⇝ $\left\{ \begin{array}{l} GW_{0,\beta}(\gamma_1,\dots,\gamma_n) \\ GW_{1,\beta} \end{array} \right. \in \mathbb{Q}$

Gopakumar–Vafa invariants ⇝ $\left\{ \begin{array}{l} n_{0,\beta}(\gamma_1,\dots,\gamma_n) \\ n_{1,\beta} \end{array} \right. \in \mathbb{Z}$ (conjectural)

Cao, Leung, Kool, Maulik, Toda, Monavari, Joyce, Gross, Oh, Thomas, ⋮

Wall-crossing ?!

Donaldson–Thomas "type" invariants ⇝ $I_{n,\beta},\ P^{t}_{n,\beta},\ M_{\beta},\ \dots \subseteq \mathcal{M}$

focus

* Heuristic by "ideal CY 4-folds"
* Ambiguity of orientations in general theory.

I. Gromov – Witten Theory

- X : CY 4-fold

- $\mathrm{vd}(\overline{M}_g(X,\beta)) = \int_\beta c_1(X) + (\dim X - 3)(1-g) = 1-g.$

$g=0$ $\quad \mathrm{vd}(\overline{M}_0(X,\beta)) = 1 \rightsquigarrow$ insertions $\gamma_1, \dots, \gamma_n \in H^*(X,\mathbb{Z})$

$$GW_{0,\beta}(\gamma_1,\dots,\gamma_n) = \int_{[\overline{M}_0(X,\beta)]^{vir}} \prod ev_i^*(\gamma_i) \in \mathbb{Q}.$$

$g=1$ $\quad \mathrm{vd}(\overline{M}_0(X,\beta)) = 0 \rightsquigarrow$ no insertions.

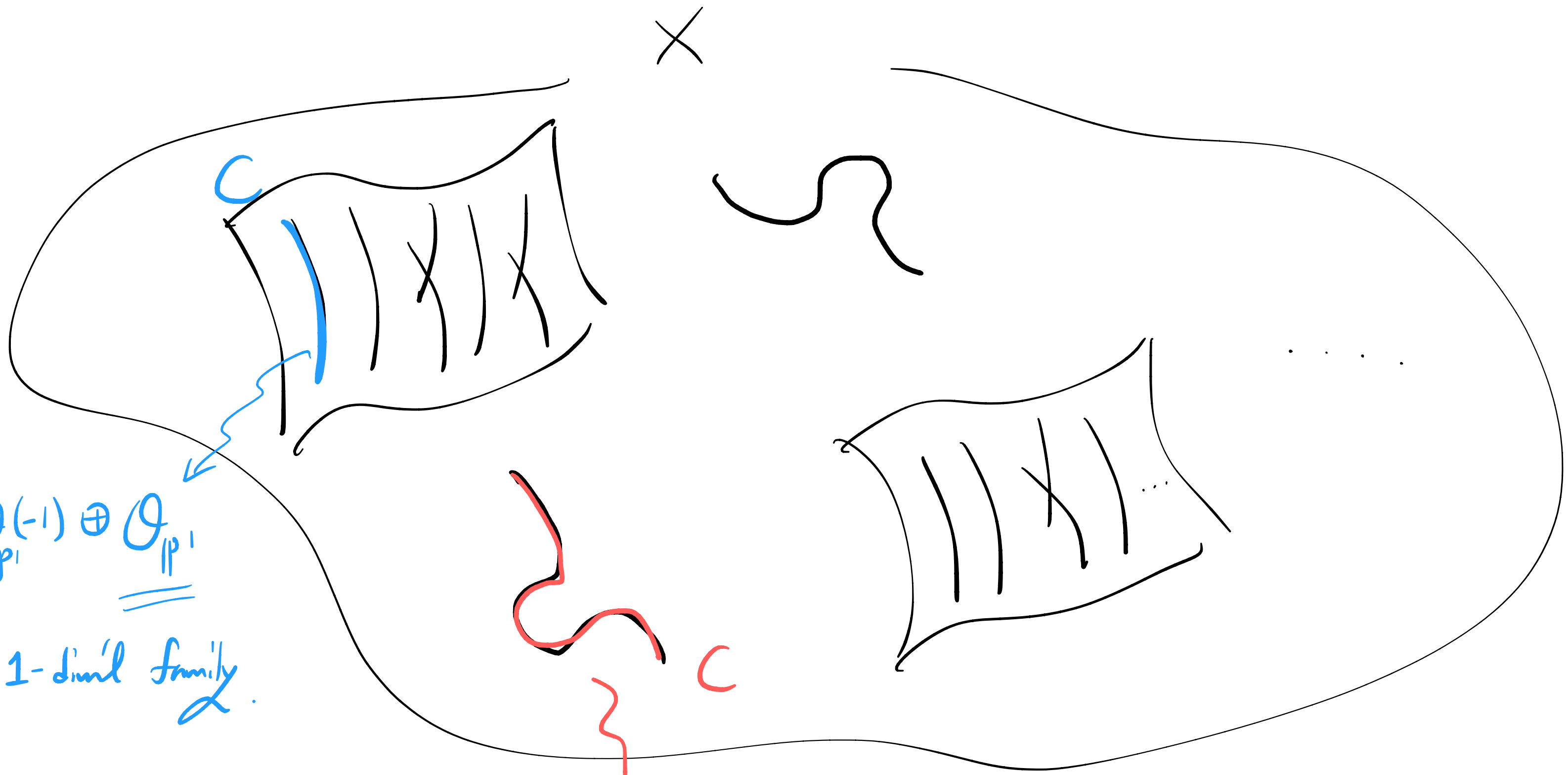
$$GW_{1,\beta} := \int_{[\overline{M}_1(X,\beta)]^{vir}} 1$$

$g\geq 2$ $\quad \mathrm{vd}(\overline{M}_0(X,\beta)) < 0 \rightsquigarrow$ trivial.

II. Ideal CY 4-folds.

On ideal CY 4-fold X,

all embedded curves live in smooth family of expected dimension.



$$N_{C/X} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \underline{\underline{\mathcal{O}_{\mathbb{P}^1}}}$$

$\therefore$ 1-dim'l family.

$$N_{C/X} \simeq L_1 \oplus L_2 \oplus L_3 \quad \text{s.t.} \quad \begin{cases} L_1 \otimes L_2 \otimes L_3 \simeq \mathcal{O}_C \\ \underset{\text{generic}}{L_i \in \mathrm{Pic}(C)} \end{cases} \Rightarrow \text{"super-rigid".}$$

# of genus 0, class β curve intersecting with $\gamma \in H^4(X, \mathbb{Z})$

$$:= n_{0,\beta}(\gamma) \in \mathbb{Z}$$

# of genus 1, class β curve

$$:= n_{1,\beta} \in \mathbb{Z}$$

Question : How to count these embedded curves?

- Physics : counts of BPS states ($\longleftrightarrow M_\beta$)
- Math : extracts from GW invariants (by Klemm – Pandharipande)

III. Gopakumar – Vafa invariants : [KP]'s proposal.

Define $n_{0,\beta}(\gamma_1,\dots,\gamma_n)$, $n_{1,\beta}$ $\in \mathbb{Q}$ as follows :

① $$\sum_{\beta>0} GW_{0,\beta}(\gamma_1,\dots,\gamma_n)\, q^\beta = \sum_{\beta>0} n_{0,\beta}(\gamma_1,\dots,\gamma_n) \sum_{d=1}^{\infty} \frac{1}{d^{3-n}} \cdot q^{d\beta}$$

multiple covers.

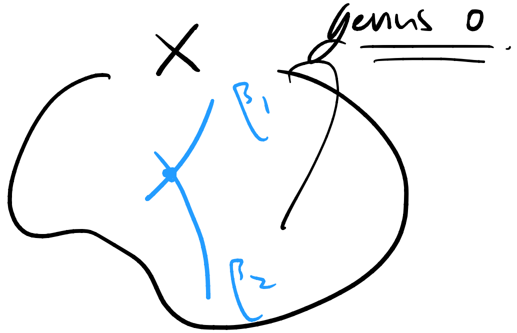
② $$\sum_{\beta>0} GW_{1,\beta}\, q^\beta = \sum_{\beta>0} n_{1,\beta} \sum_{d=1}^{\infty} \frac{\sigma(d)}{d} \cdot q^{d\beta}$$

$$+ \frac{1}{24} \sum_{\beta>0} n_{0,\beta}(c_2(T_X)) \log(1-q^\beta)$$

$$- \frac{1}{24} \sum_{\beta_1,\beta_2>0} m_{\beta_1,\beta_2} \log(1-q^{\beta_1+\beta_2})$$

lower genus contributions.

meeting invariants.



Conjecture (KP) $n_{0,\beta}(\gamma_1,\dots,\gamma_n)$, $n_{1,\beta}$ $\in \mathbb{Z}$.

On ideal CY 4-fold...

multiple cover formula :

$g=0$

$$\int_{[\overline{M}_{0,n}(\mathbb{P}^1,d)]^{vir}} e\big(-R\pi_* ev^* \underbrace{N_{\mathbb{P}^1/X}}_{\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}}\big) \cup \prod_{i=1}^{n} ev_i^*([pt]) = \frac{1}{d^{3-n}}$$

$g=1$

$$\int_{[\overline{M}_{1,0}(E,d)]^{vir}} \underbrace{e\big(-R\pi_* ev^* N_{E/X}\big)}_{1} = \frac{\sigma(d)}{d}$$

(Here, $\sigma(d) = \sum_{i|d} i$ counts # of index d sublattices in $\Lambda \simeq \mathbb{Z}^2$)

Definition of m_{β_1,β_2} using $n_{0,\beta}(\gamma)$'s for $\gamma \in H^4(X)$

* $S_1, \dots, S_\ell$ basis of $H^4(X,\mathbb{Z})$ up to torsions.

* $g_{ij} := \int_X S_i \cup S_j$. $(g^{ij}) = (g_{ij})^{-1}$.

* $\Delta_X = \cdots + \sum_{i,j} g^{ij} [S_i \otimes S_j] + \cdots \in H^8(X \times X, \mathbb{Z})$ up to torsion.

1) $m_{\beta_1,\beta_2} = m_{\beta_2,\beta_1}$

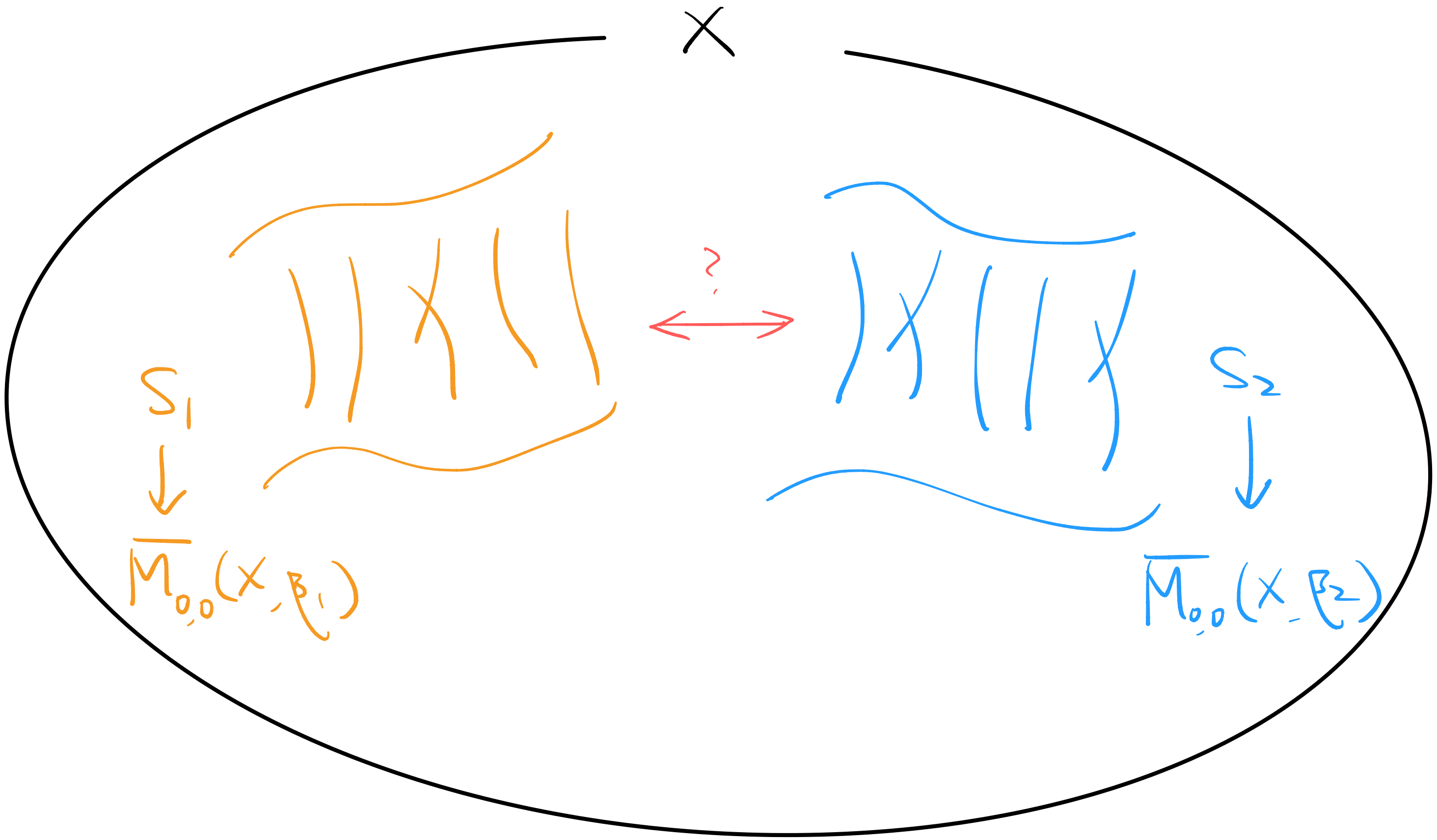
2) $\beta_1 \leq 0$ or $\beta_2 \leq 0 \Rightarrow m_{\beta_1,\beta_2} = 0$.

3) $\beta_1 \neq \beta_2 \Rightarrow m_{\beta_1,\beta_2} = \sum_{i,j} n_{0,\beta_1}(S_i)\, g^{ij}\, n_{0,\beta_2}(S_j) + m_{\beta_1,\beta_2-\beta_1} + m_{\beta_1-\beta_2,\beta_1}$.

4) $\beta_1 = \beta_2 \Rightarrow m_{\beta,\beta} = \sum_{i,j} n_{0,\beta}(S_i)\, g^{ij}\, n_{0,\beta}(S_j) + n_{0,\beta}(c_2(T_X)) - \sum_{\beta_1+\beta_2=\beta} m_{\beta_1,\beta_2}$

Heuristic behind.... 3) $m_{\beta_1,\beta_2} = \sum_{i,j} n_{0,\beta_1}(S_i)\, g^{ij}\, n_{0,\beta_2}(S_j) + m_{\beta_1,\beta_2-\beta_1} + m_{\beta_1-\beta_2,\beta_1}$

On ideal CY 4-fold...



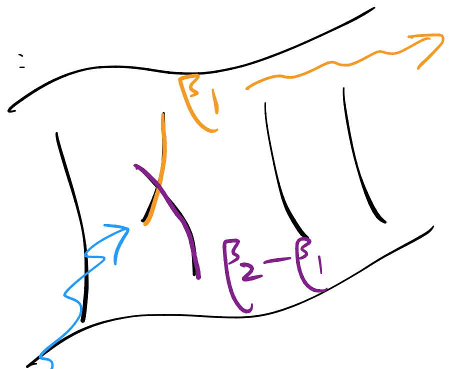
Question: How does surfaces S_1, S_2 intersect ?

Warning Not transverse even on ideal X.

$$S_1 \cap S_2 = \text{points} \quad \& \quad \text{curves}$$

$$\Rightarrow \int_X s_1 \cap s_2 = m_{\beta_1,\beta_2} + (\text{excess intersection } \#)$$

S_2 :



How does this curve "C" contribute to $\int_X s_1 \cup s_2$?

\# of such fiber $= m_{\beta_1, \beta_2 - \beta_1}$

$$0 \to \underbrace{N_{S_1/C}}_{\deg 0} \oplus \underbrace{N_{S_2/C}}_{\deg -1} \to \underbrace{N_{X/C}}_{\deg -2} \to E \to 0$$

$$\Rightarrow \mathrm{Cont}_C\left(\int_X s_1 \cup s_2\right) = \int_C c_1(E) = \boxed{-1}$$

$$\therefore \int_X s_1 \cup s_2 = m_{\beta_1,\beta_2} - m_{\beta_1-\beta_2,\beta_2} - m_{\beta_1,\beta_2-\beta_1}$$

$$\| $$

$$\sum_{i,j} n_{0,\beta_1}(S_i)\, g^{ij}\, n_{0,\beta_2}(S_j)$$ ← integral version of $\deg [\Delta_{\beta_1,\beta_2}]^{vir}$

IV. Donaldson – Thomas "type" invariants

$\mathcal{M}$:= moduli stack of $E \in D^b(\mathrm{Coh}(X))^{perf}$

s.t. $\mathrm{Ext}^{<0}(E,E) = 0$, $\det E \simeq \mathcal{O}_X$.

To be more precise, $\mathcal{M}(T) = \left\{ \begin{array}{c} \mathcal{E} \\ \downarrow \\ X \times T \end{array} \middle| \begin{array}{l} \mathcal{E} \in D^b(\mathrm{Coh}(X\times T))^{perf} \text{ s.t.} \\ \mathcal{E}|_t \text{ satisfies the conditions } \forall t \end{array} \right\}$

By [PTVV], $\mathcal{M}$ has (-2)-shifted symplectic structure.

$\Longrightarrow$ induces the same structure for open subsets !.

e.g. $P^t_n(X,\beta)$, $I_n(X,\beta)$, M_β, $M_X(v)$, $\cdots \underset{\text{open}}{\subseteq} \mathcal{M}$

scheme

$\therefore \exists\ [M]^{vir}_{o(\mathcal{L})} \in H_{2\cdot vd}(M, \mathbb{Z})$ or $A_{vd}(M) \otimes \mathbb{Z}[\tfrac{1}{2}]$.

Define $P_n(X,\beta) \ni [\mathcal{O}_X \xrightarrow{s} F]$ s.t.
- F : pure 1-dim'l
- $\mathrm{ch} F = (0, 0, 0, \beta, n)$
- $\mathrm{coker}(s)$: 0-dim'l

(Check) $\mathrm{vd}_{\mathbb{C}}(P_n(X,\beta)) = n$.

Roughly... $n \longleftrightarrow 1-g$. $(\because \chi(\mathcal{O}_C) = 1-g)$.

Define 1) $P_{0,\beta} := \int_{[P_0(X,\beta)]^{vir}} 1 \in \mathbb{Z}$

2) $P_{1,\beta}(\gamma_1, \dots, \gamma_n) := \int_{[P_1(X,\beta)]^{vir}} \prod_{i=1}^{n} \tau_0(\gamma_i) \in \mathbb{Z}$.

<u>Conjecture</u> (CMT; GV/PT correspondence)

$P_{0,0} := 1$.

① $P_{1,\beta}(\gamma_1, \dots, \gamma_n) = \sum_{\beta_0+\beta_1=\beta} P_{0,\beta_0} \cdot n_{0,\beta_1}(\gamma_1, \dots, \gamma_n)$

② $\sum_{\beta \geq 0} P_{0,\beta} \, q^{\beta} = \prod_{\beta > 0} M(q^{\beta})^{n_{1,\beta}}$ where $M(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}$.

More generally, they consider

$$P_{n,\beta}(\gamma^n) := \int_{[P_n(X,\beta)]^{vir}} \tau_0(\gamma)^n$$

and conjecture

"Wall-crossing formula"

$$①' \quad \boxed{P_{n,\beta}(\gamma^n) = \sum_{\beta_0+\beta_1+\cdots+\beta_n=\beta} P_{0,\beta_0} \cdot \prod_{i=1}^{n} n_{0,\beta_i}(\gamma).}$$

By computation,

$$①' + ② \Rightarrow PT(\exp(\gamma)) := \sum_{n,\beta} \frac{P_{n,\beta}(\gamma^n)}{n!} y^n \cdot q^\beta$$

$$\underset{\text{Conjecture}}{=} \prod_{\beta>0} \left(\exp(y q^\beta)^{n_{0,\beta}(\gamma)} \cdot M(q^\beta)^{n_{1,\beta}} \right)$$

(rational in y variable for fixed q^β)

<u>Heuristic</u> for ② : $\sum_{\beta>0} P_{0,\beta} \cdot q^{\beta} = \prod_{\beta>0} M(q^{\beta})^{n_{1,\beta}}$

Recall that for each $[\mathcal{O}_X \xrightarrow{s} F] \in P_n(X,\beta)$, we have

$$0 \longrightarrow \underbrace{\mathcal{O}_C}_{\substack{\text{pure 1-dim'l curve} \\ (\text{support of } F)}} \longrightarrow F \longrightarrow \underbrace{Q}_{\text{0-dimension}} \longrightarrow 0 \quad \Longrightarrow \quad \chi(F) \geq \chi(\mathcal{O}_C)$$

$\boxed{\text{On ideal CY 4-fold...}}$

$\chi(\mathcal{O}_C) \geq 1$ for rational curves on X. (e.g. no cuspidal α)

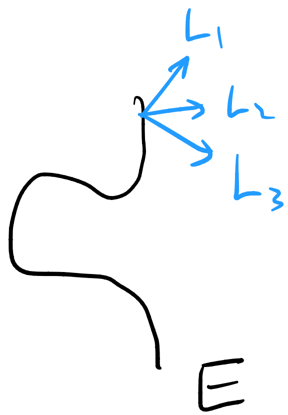
$\therefore [\mathcal{O}_X \xrightarrow{s} F] \in P_0(X,\beta)$ satisfies

- C : disjoint union of possibly non-reduced elliptic curves.
- $Q = 0$, i.e., s : surjective.

Recall that $n_{1,\beta}$ = # of (super rigid) elliptic curves on X.

$\therefore$ Enough to show : $\forall E$, $\sum_{m\geq 0} P_{0,m[E]} \cdot q^m = M(q)$

This is done by considering $X = \mathrm{Tot}_E(L_1 \oplus L_2 \oplus L_3)$, $L_i \in \mathrm{Pic}(E)$.



- $\mathbb{T} \subseteq (\mathbb{C}^\times)^3 \curvearrowright X$
- $(P_0(X, m[E]))^{\mathbb{T}} \longleftrightarrow$ plane partition λ s.t. $\lambda \vdash m$
- (Check) T^{vir}, N^{vir} trivial.

$\Rightarrow$ It follows from McMahon's identity.

Other conjecture ①' is explained next.

V. GV / PT via wall-crossing

- (X, H) : polarized CY 4-fold.
- F : 1-dimensional sheaf $\Longrightarrow \mu_H(F) := \dfrac{\chi(F)}{H.c_1(F)}$.

$\exists\ \mathbb{R}_t$-family of stability conditions for pairs.

A pair $[\mathcal{O}_X \xrightarrow{s} F]$ is Z_t-(semi) stable if

$$\begin{cases} 1)\ F : \text{pure } 1\text{-dim'l} \\ 2)\ 0 \neq \forall F' \subseteq F \Longrightarrow \mu_H(F') \underset{(\leq)}{<} t \\ 3)\ 0 \neq \forall F' \subsetneq F \Longrightarrow \mu_H(F/F') \underset{(\geq)}{>} t \\ \qquad\quad \overset{\cup\!\shortmid}{\text{image}(s)} \end{cases}$$

Rmk. 1) $t \to \infty$ recovers PT stability.

2) $\mathcal{P}_n^t(X, \beta) \longrightarrow \overline{P}_n^t(X, \beta)$: projective coarse moduli scheme.

$\exists$ finite set $W \subseteq \mathbb{R}$ (depending on (n, β)) s.t.

$t \in \mathbb{R} \setminus W \Rightarrow \nexists$ strictly Z_t-semistable pair

$\Rightarrow P_n^t(X, \beta)$: projective scheme.

Define

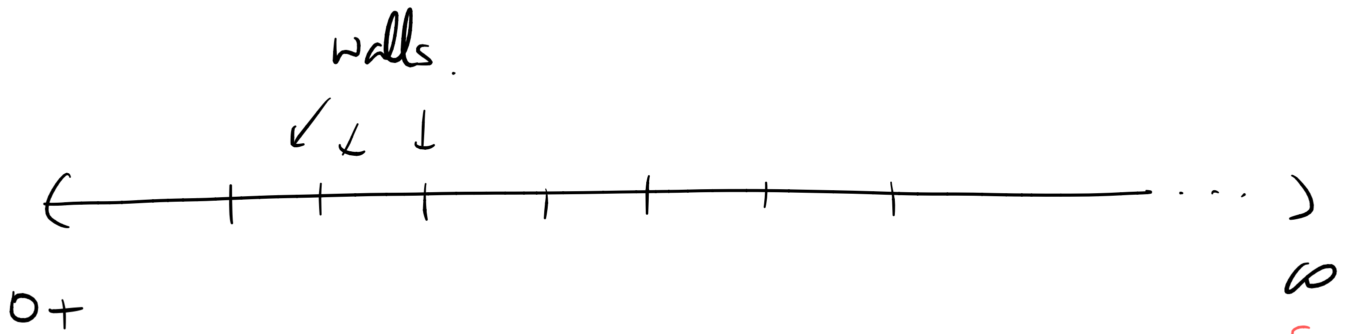
$$\begin{cases} P^t_{n,\beta}(\gamma^n) := \int_{[P^t_n(X,\beta)]^{\mathrm{vir}}} \tau_0(\gamma)^n \in \mathbb{Z} \\ PT^t(\exp(\gamma)) := \sum_{n,\beta} \frac{P^t_{n,\beta}(\gamma^n)}{n!} y^n q^\beta \end{cases}$$

<u>Conjecture</u> (CT) For $t \in \mathbb{R}_{>0} \setminus W$,

① $\quad P^t_{n,\beta}(\gamma^n) = \sum_{\substack{\beta_0+\beta_1+\cdots+\beta_n=\beta \\ \beta_i \cdot H > \frac{1}{t},\ i=1,\cdots,n}} P_{0,\beta_0} \cdot \prod_{i=1}^n n_{0,\beta_i}(\gamma)$.

This conjecture implies: $\forall\, t_0 \in \mathbb{R}_{>0}$,

$$PT^{t_0+}(\exp(\gamma)) \overset{(A)}{=} \underbrace{\prod_{\beta \text{ s.t. } \beta . H = \frac{1}{t_0}} \exp(y q^{\beta})^{n_{0,\beta}(\gamma)}}_{\text{wall-crossing term}} \cdot PT^{t_0-}(\exp(\gamma)).$$



"Conj ①"

"usual PT"

$$PT^{0+}(\exp(\gamma)) \overset{(B)}{=} \sum_{\beta \geq 0} P_{0,\beta} \cdot q^{\beta} \longleftrightarrow PT^{\infty}(\exp(\gamma)) = \sum_{n,\beta} \frac{P_{n,\beta}(\gamma^n)}{n!} y^n q^{\beta}$$

differ by wall crossing term

$*$ (A) + (B) $\Rightarrow$ GV/PT correspondence.

Heuristic argument for ①'' : $P^{t}_{n,\beta}(\gamma^n) = \sum_{\substack{\beta_0+\beta_1+\cdots+\beta_n=\beta \\ \beta_i . H > \frac{1}{t},\ i=1,\cdots,n}} P_{0,\beta_0} \cdot \prod_{i=1}^{n} n_{0,\beta_i}(\gamma)$.

On ideal CY 4-fold...

- Pick cycles $\{Z_i\}_{i=1}^n$ representing γ with generic properties:
  - * doesn't intersect elliptic curves
  - * Each rational curve C intersect at most one Z_i's.
- Denote $Q^{t}_{n}(X,\beta \mid \{Z_i\}_{i=1}^n) \subseteq P^{t}_{n}(X,\beta)$
  parametrizing pairs intersecting with all Z_i's.

$\rightsquigarrow \underline{\underline{Q^{\infty}_{1}(X,\beta_i \mid Z_i)}} = \{ C \cong \mathbb{P}^1 \text{ of class } \beta_i \text{ intersecting with } Z_i \}$.

finite of length $n_{0,\beta_i}(\gamma)$.

Claim: $$Q_n^t(X,\beta\,|\,\{z_i\}_{i=1}^n) = \coprod_{\substack{\beta_0+\beta_1+\cdots+\beta_n=\beta \\ \beta_i.H > \frac{1}{t},\ i=1,\dots,n}} P_0(X,\beta_0)\times Q_1^\infty(X,\beta_1|z_1)\times\cdots\times Q_1^\infty(X,\beta_n|z_n)$$

Pick $[\mathcal{O}_X \xrightarrow{s} F] \in \mathrm{LHS}$.

$$\Rightarrow F = \underbrace{F_0}_{\text{elliptic curves}} \oplus \bigoplus_{i=1}^n \underbrace{F_i}_{\text{rational curves meeting } z_i} \oplus \underbrace{F_{n+1}}_{\text{rational curves disjoint from } \{z_i\}\text{'s}} \quad (F_i \neq 0,\ i=1,\dots,n)$$

$$\Rightarrow n = \chi(F) = \underbrace{\chi(F_0)}_{\geq 0} + \sum_{i=1}^n \chi(F_i) + \chi(F_{n+1})$$

$\sum_{i=1}^n \chi(F_i) + \chi(F_{n+1})$: > 0 if non-zero. (uses Z_t-stability).

$$\Rightarrow \begin{cases} \chi(F_0) = 0 \Longrightarrow F_0 \simeq \mathcal{O}_{C_0} & (C_0 : \text{supp on elliptic curves}) \\ \chi(F_i) = 1 \Longrightarrow F_i \simeq \mathcal{O}_{C_i} & (C_i \simeq \mathbb{P}^1) \\ \chi(F_{n+1}) = 0 \Longrightarrow F_{n+1} = 0. \end{cases}$$

Lastly, observe that

$$\left[\mathcal{O}_X \xrightarrow{s} \bigoplus_{i=0}^{n} \mathcal{O}_{C_i} : Z_t\text{-stable} \right] \iff \left[\beta_i . H > \frac{1}{t} \quad \forall\, i = 1, \dots, n \right]$$

$\Rightarrow$: Consider $\mathcal{O}_{C_i} \subseteq \bigoplus_{j=0}^{n} \mathcal{O}_{C_j}$.

By Z_t-stability, $\mu_H(\mathcal{O}_{C_i}) = \dfrac{1}{\beta_i . H} < t$

$\Leftarrow$: Similar $\blacksquare$

$\exists$ more desirable (still heuristic) argument for

$$PT^{t_0+}(\exp(\gamma)) \overset{Ⓐ}{=} \prod_{\beta \text{ s.t. } \beta . H = \frac{1}{t_0}} \exp\left(y q^{\beta}\right)^{n_{0,\beta}(\gamma)} \cdot PT^{t_0-}(\exp(\gamma)).$$

by computing $P^{t_0+}_{n,\beta}(\gamma^n) - P^{t_0-}_{n,\beta}(\gamma^n) = \star$.

[CT], they consider simple wall $t_0 \in \mathbb{R}_{>0}$

$$
\begin{tikzcd}
P_n^{t_0+}(X,\beta) \arrow[rr, dashed, leftrightarrow] \arrow[dr, "\pi_+"'] & & P_n^{t_0-}(X,\beta) \arrow[dl, "\pi_-"] \\
& \overline{P}_n^{t_0}(X,\beta) &
\end{tikzcd}
$$

$\overline{P}_n^{t_0}(X,\beta)$: Z_{t_0}-polystable pairs.

$\in$

$I = A \oplus B$ strictly polystable where

$$
\begin{cases}
A = [\mathcal{O}_X \to F'] \in P_{n'}^{t_0}(X,\beta') \\
B = [0 \to F''] \in M_{n''}(X,\beta'')
\end{cases}
$$

They analyze locally near such $I \in \overline{P}_n^{t_0}(X,\beta)$ & uses master space to obtain formula for $P_{n,\beta}^{t_0+}(\gamma^n) - P_{n,\beta}^{t_0-}(\gamma^n)$.